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MATRIX DISPLACEMENT SOLUTION TO ELASTICA PROBLEMS OF BEAMS AND FRAMES

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Abstract—A matrix displacement approach is developed for the analysis of elastica problems of beams and frames. The displacements considered are not small in comparison with the length of the beam member. The nonlinear path is predicted by the combination use of a midpoint tangent incremental approach and coordinate transformation at every step. The formulation and procedure are simple and easy to apply. Application to a variety of examples of beams and frames shows that the approach is general and the results are reasonable.

NOTATION

EI	bending rigidity of the beam
F_x, F_y	nodal forces in the x and y directions, respectively
$[k]$	element stiffness matrix
L	length of the beam finite element
M	bending moment at the nodal point
S	axial force along the local $\bar{x}$ direction
u, v	displacement components in the x and y directions, respectively
$\bar{u}$, $\bar{v}$	displacement components in the $\bar{x}$ and $\bar{y}$ directions, respectively
x, y	reference or system coordinates
$\bar{x}$, $\bar{y}$	local coordinates for each beam finite element
α	angle of orientation of the beam finite element
Δ	incremental operator
θ	angle of rotation at the nodal point
λ, μ	direction sines and cosines of the beam finite element
ν	Poisson's ratio
$[\sigma_0]$	element incremental stiffness or initial stress matrix
$\{\ \}$	column matrix
$[\ ]$	rectangular matrix

1. INTRODUCTION

IN THE elementary beam theory, the slope of the beam is assumed small and its square is neglected in the curvature equation. The theory no longer holds when the beam is bent with large displacements.

Since the use of exact curvature formula complicates enormously the solution of the governing differential equations, studies of large displacements have been concerned mostly with single members and idealized loading and boundary conditions. The problem of cantilever beams subjected to concentrated load and uniformly distributed loads were investigated by Barton [1], Bisshopp and Drucker [2], Rhode [3], Scott and Carver [4].

Simply supported beams were studied by Conway [5], Scott and Carver [4], Gospodnetic [6]. Circular beams were studied by Seames and Conway [7] and Mitchell [8]. Numerical analysis for such problems were performed by Iyengar [9], Seames and Conway [7], Wang, Lee and Zienkiewicz [10] and Wang [11].

The extension of the above development to the solution of frame problems are, however, limited to a few specially idealized cases. Kerr [12] studied a square frame loaded at the midpoints of two opposite sides. Jenkins, Seitz and Przemieniecki [13] analyzed a diamond-shaped frame loaded diagonally at two corners.

Because of the versatile nature of the finite element method, it seems desirable to develop a finite element procedure for solving the large displacement problems of beams and frames so that the frames with arbitrary geometry, complex loading and boundary conditions can be treated straightforwardly. Tada and Lee [14] developed a beam finite element and demonstrated a problem of a cantilever beam loaded at the free end. Instead of using the conventional displacements and rotations as nodal degrees of freedom, they adopted nodal coordinates and direction cosines of a tangent vector in the deformed configuration. The direction sines and cosines were expanded into Taylor series. The stiffness matrices were obtained from the nonlinear equation of equilibrium by using Galerkin's method. The iterative solution procedure was started with the computation of 16 coefficients in the interpolation function and the assumption of all the nodal rotations. In their example of the cantilever beam, 20 elements were used. Results were in good agreement with those given by Bisshopp and Drucker [2].

Aiming at a simpler formulation and procedure, also at less use of elements, a straightforward matrix displacement finite element procedure is developed here. The conventional stiffness formulation for small deflection is used. The effect of axial force is taken into account by adding incremental stiffness or initial stress matrix to the stiffness equation. The nonlinear load–displacement path is predicted by a linearized midpoint tangent incremental procedure together with coordinate transformation at every step. Application to examples of beams and frames demonstrates that the method is simple yet general and the results are in reasonable agreement with alternative known solutions.

2. STIFFNESS FORMULATION

The present development is based on the assumption that the material is linearly elastic and the displacements are not small in comparison with the length of the beam.

The solution procedure includes first formulating the stiffness equations for a beam element based on the small deflection theory but with the inclusion of the effect of axial force, then applying a linearized midpoint tangent incremental approach and coordinate transformation at every step. If the displacements obtained at every step are small with reference to the local coordinates such that the squares of the slope-increment are negligible in comparison with unity, the small deflection theory should hold.

(a) *The stiffness formulation for small deflection with respect to local coordinates*

In the small deflection theory, the strain energy for an extensible beam segment shown in Fig. 1 is given by

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}\right)^2 \mathrm{d}\bar{x} + \frac{1}{2}\int_0^L EA\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 \mathrm{d}\bar{x} \tag{1}$$

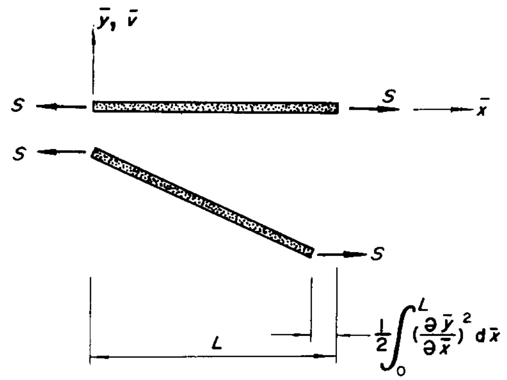

FIG. 1. An inextensible beam segment with an axial force S.

where $\bar{x}$ and $\bar{y}$ are the local coordinates and $\bar{u}$ and $\bar{v}$ are the axial and transverse displacements, respectively.

If an initial axial force S acts in the local $\bar{x}$ direction, the strain energy due to small bending becomes

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}\right)^2 \mathrm{d}\bar{x} + \frac{1}{2}\int_0^L EA\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 \mathrm{d}\bar{x} + \frac{S}{2}\int_0^L \left(\frac{\partial \bar{v}}{\partial \bar{x}}\right) \mathrm{d}\bar{x}. \tag{2}$$

Since the curvature expression used in equation (2) is not exact, the equation is valid only when the deflection is small with reference to the local coordinates $\bar{x}$ and $\bar{y}$.

The displacement patterns for $\bar{u}$ and $\bar{v}$ may be assumed, for small deflection, as

$$\bar{u}(x) = \left(1-\frac{\bar{x}}{L}\right)u_1 + \left(\frac{\bar{x}}{L}\right)u_2$$

$$\bar{v}(x) = \left[1+2\left(\frac{\bar{x}}{L}\right)^3 - 3\left(\frac{\bar{x}}{L}\right)^2\right]\bar{v}_1 + \left[3\left(\frac{\bar{x}}{L}\right)^2 - 2\left(\frac{\bar{x}}{L}\right)^3\right]\bar{v}_2$$
$$+\left[\bar{x}-2\frac{\bar{x}^2}{L}+\frac{\bar{x}^3}{L^2}\right]\theta_1 + \left[-\frac{\bar{x}^2}{L}+\frac{\bar{x}^3}{L^2}\right]\theta_2 \tag{3}$$

where $\bar{u}_1$, $\bar{u}_2$, $\bar{v}_1$, $\bar{v}_2$, θ_1 and θ_2 are the nodal displacements and rotations shown in Fig. 2.

Substituting equation (3) in (2) and performing partial differentiation with respect to

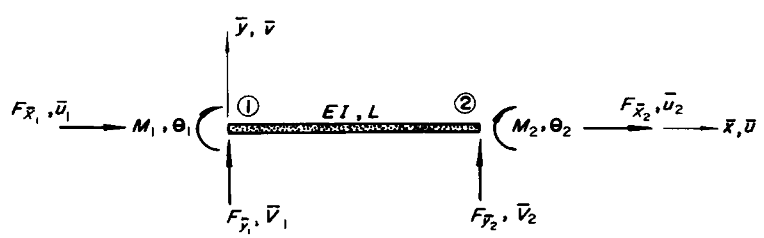

FIG. 2. Assumed degrees of freedom of a beam finite element.

each of the six nodal degrees of freedom, the nodal force–displacement relationship is obtained following the Castigliano's theorem,

$$\begin{Bmatrix} F_{\bar{x}_1} \\ F_{\bar{y}_1} \\ M_1 \\ F_{\bar{x}_2} \\ F_{\bar{y}_2} \\ M_2 \end{Bmatrix} = \left[E \begin{bmatrix} \frac{A}{L} & & & & & \\ 0 & \frac{12I}{L^3} & & \text{Symmetric} & & \\ 0 & \frac{-6I}{L^2} & \frac{4I}{L} & & & \\ -\frac{A}{L} & 0 & 0 & \frac{A}{L} & & \\ 0 & \frac{-12I}{L^3} & \frac{6I}{L^2} & 0 & \frac{12I}{L^3} & \\ 0 & \frac{-6I}{L^2} & \frac{2I}{L} & 0 & \frac{6I}{L^2} & \frac{4I}{L} \end{bmatrix} + \frac{S}{10L} \begin{bmatrix} 0 & & & & & \\ 0 & 12 & & \text{Symmetric} & & \\ 0 & -L & \frac{4L^2}{3} & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -12 & L & 0 & 12 & \\ 0 & -L & \frac{-L^2}{3} & 0 & L & \frac{4L^2}{3} \end{bmatrix} \right] \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \theta_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} \tag{4a}$$

or symbolically

$$\{\bar{p}\} = [[\bar{k}] + [\bar{\sigma}_0]]\{\bar{q}\} \tag{4b}$$

where $\{\bar{p}\}$ and $\{\bar{q}\}$ are the force and displacement vectors at the nodes, respectively; $[\bar{k}]$ is the stiffness matrix; and $[\bar{\sigma}_0]$ is the initial stress or incremental stiffness matrix. Equation (4a) was formulated previously by Gallagher and Padlog [15] for the solution of Euler buckling load of a column.

(b) *Coordinate transformation*

In the present incremental procedure, local coordinates are chosen to represent the individual element at every step. Each element formulation is then transformed from the local coordinates to a set of convenient reference coordinates. Therefore, the applied loads, boundary conditions, and the displacements are uniquely expressed in terms of the reference coordinates.

Figure 3 shows an arbitrarily oriented beam element and the two coordinate systems. The nodal displacements in terms of local coordinates and reference coordinates are related by the equation that

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \theta_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \theta_2 \end{Bmatrix} = \left[\begin{array}{ccc:ccc} \lambda & \mu & 0 & & & \\ -\mu & \lambda & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hdashline & & & \lambda & \mu & 0 \\ & 0 & & -\mu & \lambda & 0 \\ & & & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{5a}$$

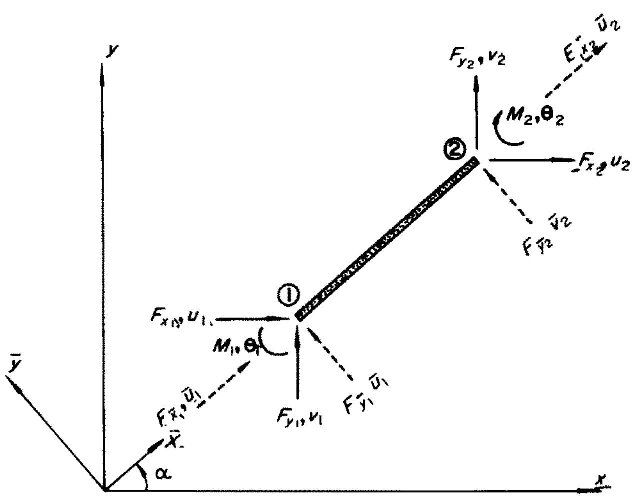

FIG. 3. Transformation from local ($\bar{x}$, $\bar{y}$) to reference (x, y) coordinates for a beam element.

or symbolically

$$\{\bar{q}\} = [T]\{q\} \tag{5b}$$

where $\lambda = \cos\alpha$ and $\mu = \sin\alpha$ with α being the angle of orientation of the beam as shown in Fig. 3.

Because of the orthogonal property of the transformation matrix, it can be shown that [16]

$$\underset{6\times 6}{[k]} = \underset{6\times 6}{[T]^T}\ \underset{6\times 6}{[\bar{k}]}\ \underset{6\times 6}{[T]} \tag{6}$$

where $[k]$ is the transformed stiffness matrix of the element in terms of reference coordinates.

Applying the transformation process as given by equation (6) to both the stiffness and the incremental stiffness or initial stress matrix given by equation (4), the element force–displacement relationship is obtained in terms of the reference coordinates

$$
\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ \theta_1 \\ \hline F_{x_2} \\ F_{y_2} \\ \theta_2 \end{Bmatrix} = \left[\frac{E}{L} \left[\begin{array}{ccc|ccc}
A\lambda^2+\frac{12I}{L^2}\mu^2 & & & & \text{Symmetric} & \\
\left(A-\frac{12I}{L^2}\right)\lambda\mu & A\mu^2+\frac{12I}{L^2}\lambda^2 & & & & \\
\frac{6I}{L}\mu & -\frac{6I}{L}\lambda & 4I & & & \\
\hline
-\left(A\lambda^2+\frac{12I}{L^2}\mu^2\right) & -\left(A-\frac{12I}{L^2}\right)\lambda\mu & -\frac{6I}{L}\mu & A\lambda^2+\frac{12I}{L^2}\mu^2 & & \\
-\left(A-\frac{12I}{L^2}\right)\lambda\mu & -\left(A\mu^2+\frac{12I}{L^2}\lambda^2\right) & \frac{6I}{L}\lambda & \left(A-\frac{12I}{L^2}\right)\lambda\mu & A\mu^2+\frac{12I}{L^2}\lambda^2 & \\
\frac{6I}{L}\mu & -\frac{6I}{L}\lambda & 2I & -\frac{6I}{L}\mu & \frac{6I}{L}\lambda & 4I
\end{array} \right] \right.
$$

$$
\left. + \frac{S}{10L} \left[\begin{array}{ccc|ccc}
12\mu^2 & & & \text{Symmetric} & & \\
-12\lambda\mu & 12\lambda^2 & & & & \\
L\mu & -L\lambda & \frac{4L^2}{3} & & & \\
\hline
-12\mu^2 & 12\lambda\mu & -L\mu & 12\mu^2 & & \\
12\lambda\mu & -12\lambda^2 & L\lambda & -12\lambda\mu & 12\lambda^2 & \\
L\mu & -L\lambda & -L^2/3 & -L\mu & L\lambda & \frac{4L^2}{3}
\end{array} \right] \right] \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ \hline u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{7a}
$$

or symbolically

$$\{p\} = [[k]+[\sigma_0]]\{q\} \tag{7b}$$

In the case of inextensible bar, the axial rigidity EA can be assigned a very large value in comparison with the flexural rigidity EI.

The overall stiffness equation obtained after the assemblage of individual element formulations is presented here by using capital lettered symbols

$$\{P\} = [[K]+[\Sigma_0]]\{Q\}. \tag{8}$$

(c) *Midpoint tangent incremental procedure*

The linearized incremental formulation can be obtained by applying an incremental operator Δ to equation (8),

$$\{\Delta P_i\} = [[K]_{i-1}+[\Sigma_0]_{i-1}]\{\Delta Q_i\} \tag{9}$$

or

$$\{\Delta Q_i\} = [[K]_{i-1}+[\Sigma_0]_{i-1}]^{-1}\{\Delta P_i\} \tag{10}$$

where i is the step number. The axial force S in each element incremental stiffness or initial stress matrix $[\sigma_0]$ is obtained as

$$S = F_x \cos\alpha + F_y \sin\alpha. \tag{11}$$

The angle α which defines the orientation of the straight beam element may be found by setting

$$\tan\alpha = \frac{y_2-y_1}{x_2-x_1} \tag{12}$$

where x and y are the reference coordinates of the two nodal points (subscripted as 1 and 2) found at the end of step $i-1$. The nodal forces F_x and F_y are calculated from the element stiffness matrix $[k]$ and the nodal displacements. It is seen in equation (4) that F_x and F_y found at both nodal points are always equal in magnitude but opposite in sign. If the structure considered is determinate, F_x and F_y can then be obtained from statics without even using equation (4a).

One of the conveniences of the matrix displacement formulation as given by equation (9) is that the given condition can be either load increments, or displacement increments, or even the combination of both. Such conditions can be handled by rearranging and partitioning the matrices and then dealing with the submatrix equations [16].

In equation (9) or (10), both the stiffness matrix $[K]$ and the initial stress matrix $[\Sigma_0]$ used in step i are based on the direction sines μ and cosines λ calculated at step $i-1$. Furthermore, the initial stress matrix $[\Sigma_0]$ varies linearly with respect to the element axial forces S which also depend upon the displaced configuration at the end of previous step. If the nodal displacements $\{Q_i\}$ at the end of step i can be assumed previously, a better approximation can be achieved by using the values of S, λ and μ based on the average of the displacements at the two steps. It is suggested here that an estimate may be made that

$$\{\Delta Q_i\} = \beta\{\Delta Q_{i-1}\} \tag{13}$$

where β is the ratio between the load increment at step i and that at step $i-1$. The procedure thus described is termed here as the midpoint tangent incremental approach.

It is interesting to examine the error introduced by the midpoint tangent incremental approach. Figure 4 shows a load–displacement curve. The load and displacement are denoted by ξ and η, respectively. The true displacement vector at the end of step i may be expressed by a four-term Taylor series

$$\eta(\xi+\Delta\xi) = \eta(\xi)+\Delta\xi\eta'(\xi)+\tfrac{1}{2}\Delta\xi^2\eta''(\xi)+\tfrac{1}{6}\Delta\xi^3\eta'''(\xi). \tag{14}$$

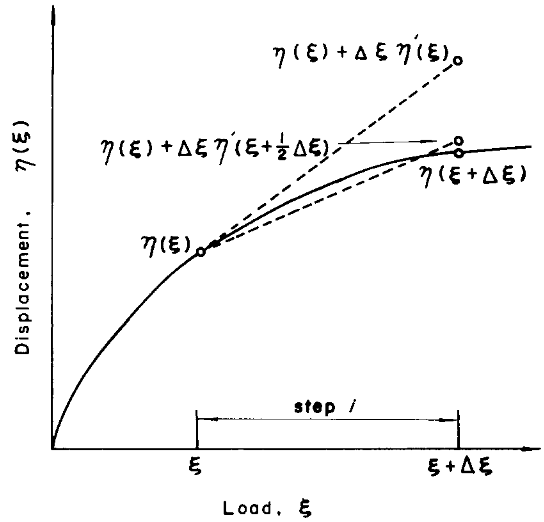

FIG. 4. Errors of an ordinary incremental step and a midpoint tangent incremental step.

This indicates that the ordinary linear incremental approach induces in each increment an error whose dominant terms are (see Fig. 4)

$$\varepsilon = -\tfrac{1}{2}\Delta\xi^2\eta''(\xi)-\tfrac{1}{6}\Delta\xi^3\eta'''(\xi). \tag{15}$$

If the midpoint tangent incremental procedure is applied, the displacement vector at the end of step i becomes

$$\eta(\xi)+\Delta\xi\eta'(\xi+\tfrac{1}{2}\Delta\xi) = \eta(\xi)+\Delta\xi\eta'(\xi)+\tfrac{1}{2}\Delta\xi^2\eta''(\xi)+\tfrac{1}{8}\Delta\xi^3\eta'''(\xi)+\cdots \tag{16}$$

Comparing equations (16) and (14), the dominant term in error is seen to be

$$\varepsilon = \frac{-1}{24}\Delta\xi^3\eta'''(\xi). \tag{17}$$

An investigation of the errors given by equations (15) and (17) reveals that the midpoint tangent incremental approach certainly results in less error than that obtained from the ordinary incremental approach. A discussion of the first error term in equation (15) was given in Ref. [17].

3. RESULTS AND EVALUATIONS

The formulation and procedure developed above have been applied to four problems for which the alternative exact or approximate solutions are available for evaluation.

(a) *A cantilever beam with a concentrated load at the free end*

A cantilever beam, subjected at the free end to a concentrated load, was analyzed with the results for vertical and horizontal displacements of the free end shown in Fig. 5. Four finite elements were used. The size of increment at each step was progressively enlarged in accordance with a geometric series. These sizes are indicated by small dots in Fig. 5. The employment of unequal load increments according to a geometric series was suggested in Ref. [17].

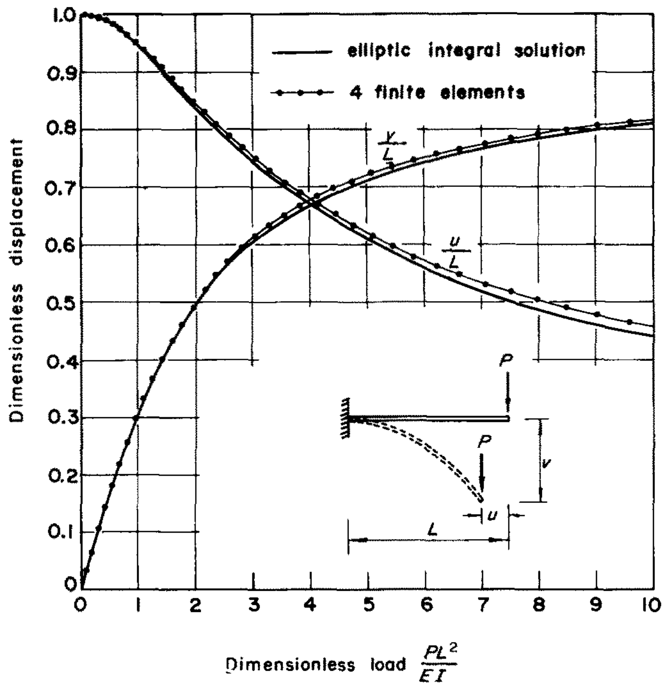

FIG. 5. Horizontal and vertical components of the free end displacements for a cantilever beam with a concentrated load.

The results from an exact analytical solution in terms of elliptic integrals [2] are also shown in Fig. 5 for comparison. Good agreement is indicated.

(b) *A cantilever beam with uniformly distributed load*

A cantilever beam subjected to uniformly distributed load was analyzed with the results for vertical and horizontal displacements of the free end shown in Fig. 6. Again, four finite elements were used. The sizes of the increment are indicated in the figure by small dots.

For a cantilever beam subjected to uniformly distributed load, the differential equations do not lead to any direct solution. Only approximate solutions are possible. Rhode [3] expanded the slope term in a power series of the arc length to analyze the problem approximately. The results are shown in Fig. 6 for comparison. The agreement is reasonable.

(c) *A diamond-shaped frame loaded diagonally at two joints*

A diamond-shaped frame loaded by forces applied at a pair of diagonally opposite joints is shown in Fig. 7. The two loaded joints are assumed to be hinged while the two free joints are assumed to be rigid. This problem is similar to finding the deflection of a cantilever beam loaded by an oblique (45°) concentrated load at the free end. It was considered by Frish–Fay [9]. It was also studied extensively by Jenkins, Seitz and Przemieniecki [13]. Jenkins *et al.* provided an analytical solution which was in good agreement with their experimental results. Their results for the dimensionless horizontal elongation and vertical contraction of the diagonals are shown in Figs. 7 and 8, respectively.

Because of symmetry, only a quadrant of the frame need be analyzed by the present finite element approach and four elements were used. The results are shown in Figs. 7

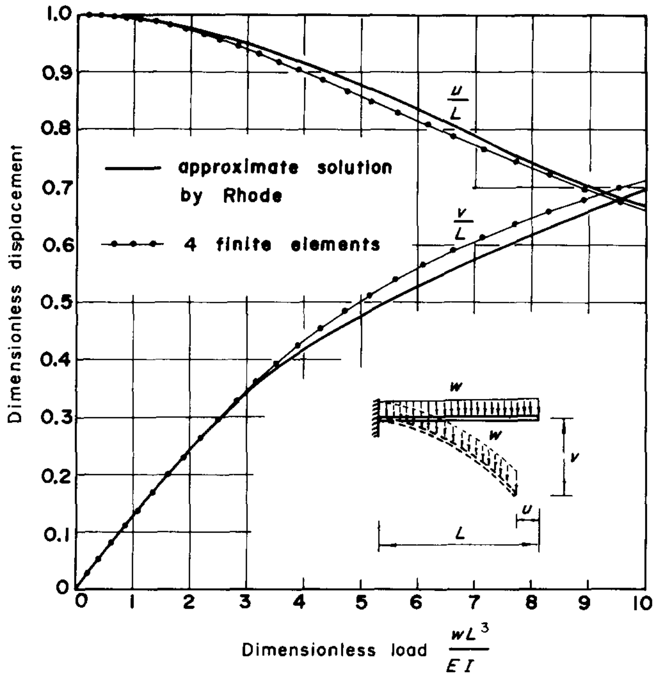

FIG. 6. Horizontal and vertical components of the free end displacements for a cantilever beam with uniformly distributed loads.

and 8 where the dots indicate the increment sizes. It is seen that the present results are in good agreement with those obtained analytically.

The displacement configuration at several loading stages were given in Ref. [13]. The same shapes were also obtained in this study.

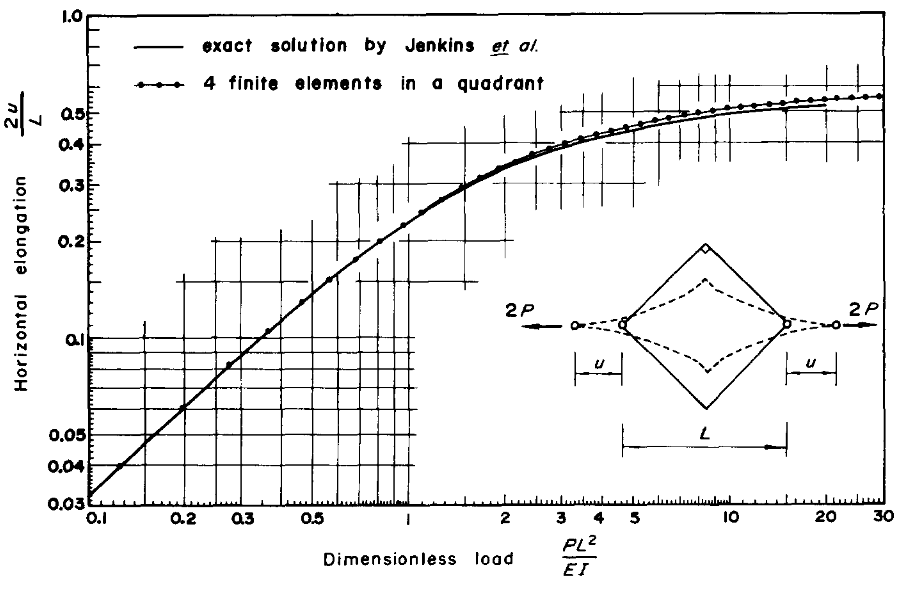

FIG. 7. Horizontal elongation of a diamond-shaped frame.

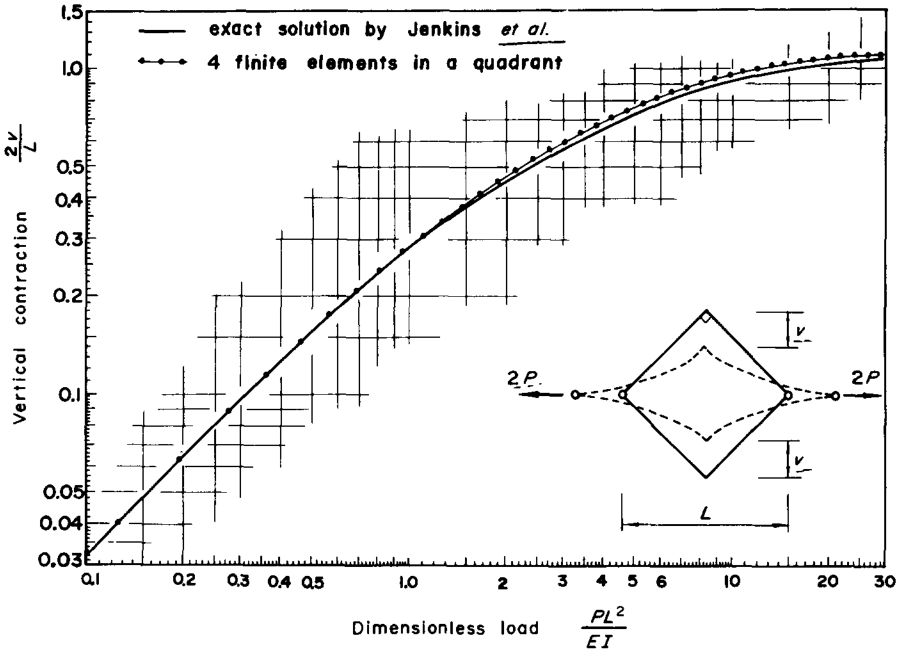

FIG. 8. Vertical contraction of a diamond-shaped frame.

(d) *A square frame loaded at midpoints of two opposite sides*

The fourth example considered was a square frame subjected to two forces applied at the midpoints of a pair of opposite sides (see Fig. 9). An analytical solution and an experiment were provided for this problem by Kerr [12]. His results for the dimensionless horizontal elongation and vertical contraction of the length between the midpoints of two opposite sides are shown in Figs. 9 and 10 respectively.

Because of symmetry, only a quadrant of the frame need by analyzed by the present finite element method and eight elements were used. The results are shown in Figs. 9 and 10 where the dots indicate the increment sizes. The agreement among the finite element solution, the analytical solution, and the experimental solution shows that the present approach is reasonable.

Typical deformed shapes for the square frame are shown in Fig. 11 for four different loading stages.

4. CONCLUDING REMARKS

A beam finite element stiffness formulation which includes the effect of axial force and coordinate transformation has been developed. The nonlinear path can be predicted by a midpoint tangent incremental approach and coordinate transformation at every step. The procedure is general yet simple to follow.

Because of the complexity of the nonlinear differential equations, analytical solutions for the beam and frame problems may be obtained only for a few idealized cases. However, four special cases have been chosen to evaluate the present formulation and procedure. The present method has been shown to give reasonable solutions.

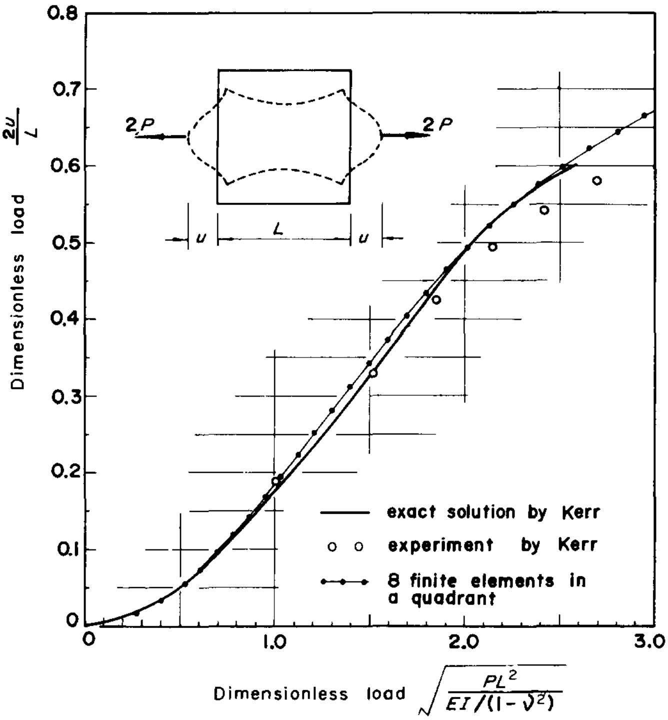

FIG. 9. Horizontal elongation of a square frame.

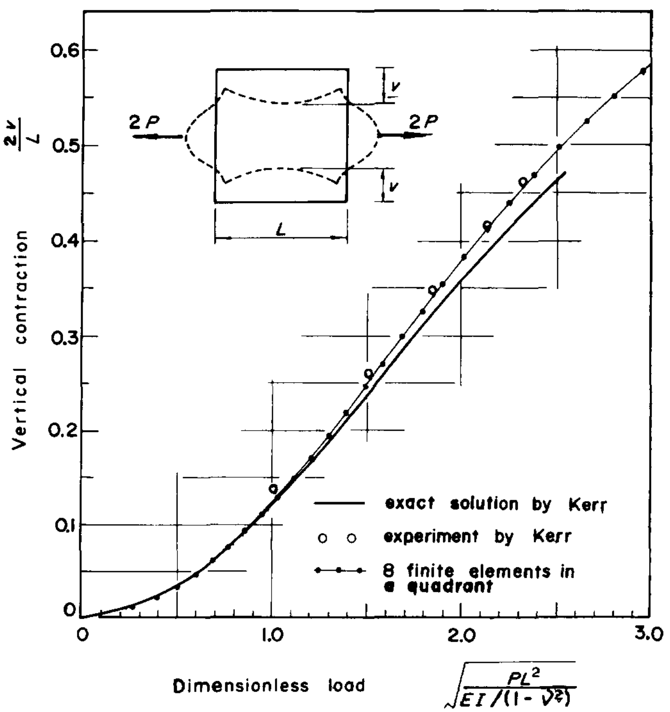

FIG. 10. Vertical contraction of a square frame.

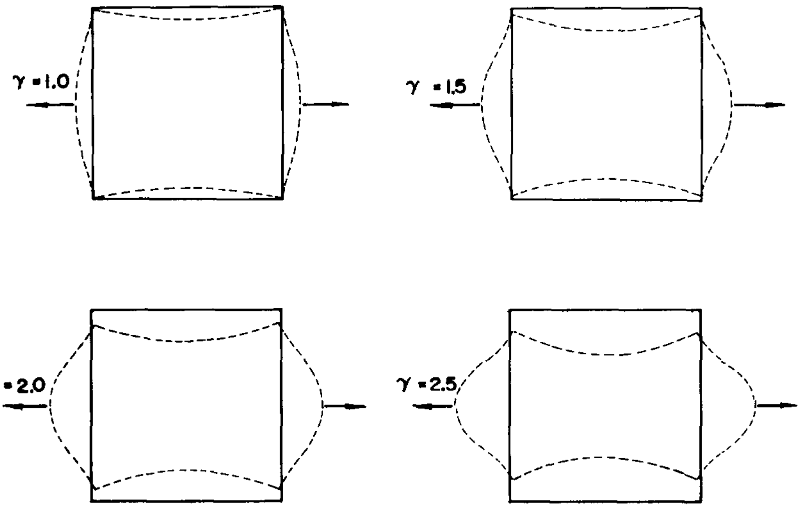

FIG. 11. Deformed shapes of a square frame for different values of loading parameter $\gamma = PL^2/\sqrt{[EI(1 - \nu^2)]}$ (displacements and dimensions of the frame are in the same scale).

With this development, the general large displacement problems of beams and frames can be solved. The geometry, the loading, and the boundary conditions for such structures can be completely arbitrary.

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Абстракт—Разрабатывается подход в виде матрицы перемещений, для анализа упругих балок и рам. Рассматриваемые перемещения не являются малыми по сравнению с длиной балки. Предсказывается нелинейная путьрасчета, на основе комбинации использования подхода постепенно нарастяющей средней точки касательной и преобразования координат для каждого этапа. Формулировка и методика несложны и легко применимы. Применение к ряду примеров балок и рам указывает общность метода и умеренные результаты.